α-MATHEMATICS

Alpha Mathematics
FINAL EXAMINATION PAPER

November 3, 2020 **Time: 3 hours**

Grade 12 **Total: 200 marks**

INSTRUCTIONS AND INFORMATION

Carefully read through the following instructions before answering the question paper:

1. Answer all 10 questions on this exam paper.
2. Write your name and ID number on the front page of the question paper.
3. Non-programmable calculators may be used, unless otherwise indicated at a specific question.
4. Unless indicated otherwise, all answers, where applicable, must be given correct to two decimal places.
5. The diagrams in the question paper are not necessarily drawn to scale.
6. All angles are given in radians. Answers must be given in radians where applicable.
7. This question paper consists of a front page, 26 pages and a formula sheet of 3 pages.
8. Question 1 consists of 10 multiple choice questions. Answer it on the answer sheet. This answer sheet is at the front of the paper.
   Do not remove the answer sheet from the question paper.
9. For all other questions, all necessary calculations must be shown clearly. The correct answer on its own will not necessarily lead to full marks.
10. Additional writing space is provided at the end of this question paper. Clearly indicate if you made use of this to complete a question.
11. Write neatly and legibly.

QUESTION 1 [20 MARKS]

- Answer this question **on the answer sheet** that is attached to the front, by marking an X (cross) on A, B, C or D.
- Please **DO NOT** remove this page from the question paper.
- Each question counts 2 marks.

1.1 Given $f(x) = \sqrt{5x}$. Then $f'(5) =$

(A) $\sqrt{5}$ **(B)** $\frac{1}{2}$

(C) $\frac{\sqrt{5}}{2}$ **(D)** $\frac{1}{10}$

1.2 The expansion of the power series $\left(1 - \frac{x}{2}\right)^{-5}$ will be valid if

(A) $|x| < 2$ **(B)** $|x| < \frac{1}{2}$

(C) $|x| > 2$ **(D)** $|x| > \frac{1}{2}$

1.3 The graph of $y = -|2x - 6| - 4$ has a salient point at

(A) $(3;\ 4)$ **(B)** $(6;\ -4)$

(C) $(3;\ -4)$ **(D)** $(-3;\ -4)$

1.4 The function $f(x) = \begin{cases} cx^2 + 2x & if \quad x < 2 \\ 2x + 4 & if \quad x \geq 2 \end{cases}$ is continuous. The value of $c =$

(A) -1 **(B)** -2 **(C)** 2 **(D)** 1

1.5 The size of the vector $2i - 3j + ak$ is $\sqrt{14}$. Then a possible value of $a =$

(A) 2 **(B)** 3 **(C)** -1 **(D)** -4

1.6 Which of the following statements is always true?

(i) If $\lim_{x \to a^+} f(x)$ exists, then f is continuous in $x = a$.

(ii) If $\lim_{x \to a} f(x) = f(a)$, then f is differentiable in $x = a$.

(iii) If f is differentiable in $x = a$, then $\lim_{x \to a} f(x) = f(a)$.

(A) Only (i) **(B)** Only (ii) **(C)** Only (iii) **(D)** Only (i) and (iii)

1.7 The gradient of the tangent to the graph $\sin x + \cos y = \sqrt{3}$ at the point $(\frac{\pi}{3}; \frac{\pi}{6})$ is equal to

(A) 1 **(B)** -1 **(C)** $\frac{1}{\sqrt{3}}$ **(D)** $\sqrt{3}$

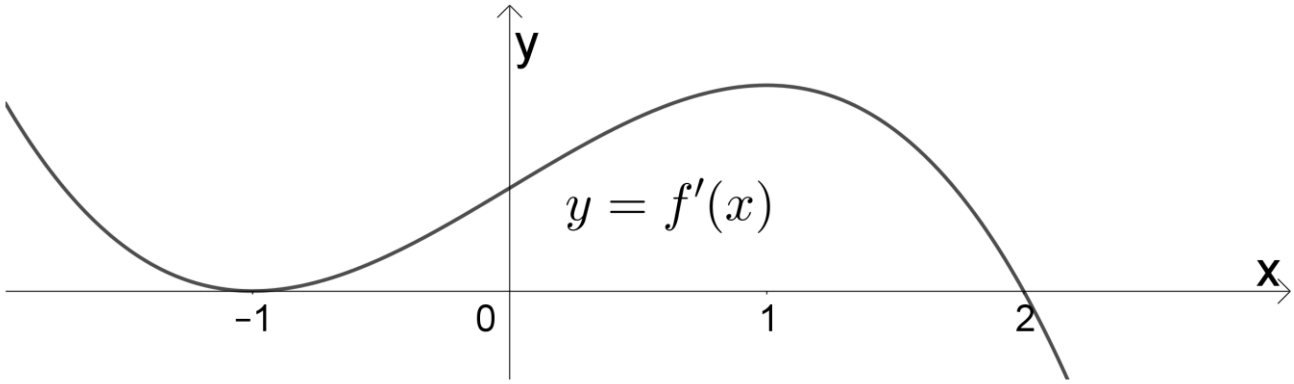
1.8 $\int_0^a e^x \, dx = 2$, then $a =$

(A) $ln2$ **(B)** $ln3$ **(C)** 2 **(D)** 3

1.9 The position function of a particle is given by $S(t) = 6 + 3t^2 - t^3, t \geq 0$.
When will the particle have no acceleration? At $t =$

(A) -1 **(B)** 2 **(C)** 0 **(D)** 1

1.10 The graph of f', the derivative of f is shown beneath.



Which one of the following statements is **not** true: The graph of $y = f(x)$

(A) is concave up at $x = 0$.

(B) has a stationary point at $x = -1$.

(C) has a point of inflection at $x = -1$.

(D) has a local minimum at $x = 2$.

Answer the following questions **on the exam paper** on the lines provided at the end of each question. Clearly indicate if you use the additional writing space at the end of the paper to complete a question.

QUESTION 2 [18 MARKS]

2.1 Solve for x: $e^{\frac{x}{2}+10} = 50$ (3)

2.2 The magnitude of an earth quake can, according to the Richter scale, be determined with the equation

$$M_n = \log\frac{T_n}{k}$$

where M_n is the magnitude, T_n the intensity of the earth quake at place n and k a constant. In 1960 there was an earth quake in Chile with a magnitude of M_C = 9,5.

(a) Determine T_C the intensity of the earth quake in Chile in terms of k. (2)

(b) In San Francisco there was an earth quake in 1989 with a magnitude of M_S = 6,5. Determine how much more intensive (thus $\frac{T_C}{T_S}$) the earth quake was in Chile. (3)

2.3 Solve for x: $|x-2| = x^2$ (5)

2.4 Given the set of equations:

$$\begin{aligned} x + 3y + kz &= 4 \\ 4x - 2y - 10z &= -5 \\ x + y + 2z &= 1 \end{aligned}$$

Use Cramer's rule and determine the value of k for which the set has no unique solution. (5)

QUESTION 3 [18 MARKS]

3.1 Use de Moivre's theorem and simplify $\frac{\left(1-\sqrt{3}i\right)^4}{(-1+i)^{12}}$

Give all angles in terms of π and use surd form if necessary. Do all the calculations in polar or exponential form, but give the final answer in rectangular form and use surds if necessary. (8)

3.2 Use a Riemann sum and calculate the value of $\int_0^3 (3x^2 + 2x - 1)dx$. (10)

QUESTION 4 [20 MARKS]

4.1 Given $u = (2;\ -1; 1)$ and $v = -i + 2j - 3k$.

(a) (i) Determine the magnitude of vector $\boldsymbol{u}$. (1)

(ii) Determine the dot product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. (2)

(iii) Caculate the angle between the two vectors. (3)

(b) Calculate the area of the parallelogram formed between the two vectors. (6)

4.2 The equation $x^4 + 5x^3 + 35x^2 + 37x + 58 = 0$ has a root $x = -2 - 5i$.
Determine the real root(s) of the equation, if any. (8)

QUESTION 5 [19 MARKS]

5.1 Use the binomial theorem and determine the sixth term in the expansion of

$$\left(\frac{x^2}{2} - \frac{2}{x}\right)^{11}$$

(6)

5.2 Determine and simplify the first three terms of the power series

$$\frac{1}{\sqrt{1-2x}}$$

(5)

5.3 In an exam, the following question is asked:
Use mathematical induction and prove that

$$\frac{1}{4\times 1^2-1}+\frac{1}{4\times 2^2-1}+\frac{1}{4\times 3^2-1}+\cdots+\frac{1}{4\times n^2-1}=\frac{n}{2n+1}$$

A learner does the following:

Let $n=1$: LHS = $\frac{1}{3}$ and RHS = $\frac{1}{2.1+1}=\frac{1}{3}$, thus true for n=1.

A **Let $n=k$:**

$$\frac{1}{4\times 1^2-1}+\frac{1}{4\times 2^2-1}+\frac{1}{4\times 3^2-1}+\cdots+\frac{1}{4\times k^2-1}=\frac{k}{2k+1}$$

B **Let $n=k+1$:**

RHS = $\frac{k+1}{2(k+1)+1}=\frac{k+1}{2k+3}$

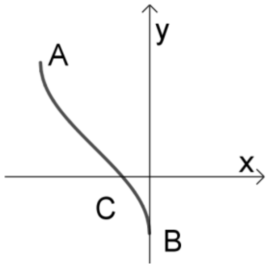
LHS = $\frac{1}{4\times 1^2-1}+\frac{1}{4\times 2^2-1}+\frac{1}{4\times 3^2-1}+\cdots+\frac{1}{4\times (k+1)^2-1}=\cdots$

(a) The learner makes a mistake in step A. Correct it. (1)

(b) The learner does not know how to do the next steps in B. Complete the steps to make the statement true for all $n \in \mathbb{N}$ according to the principle of mathematical induction. (7)

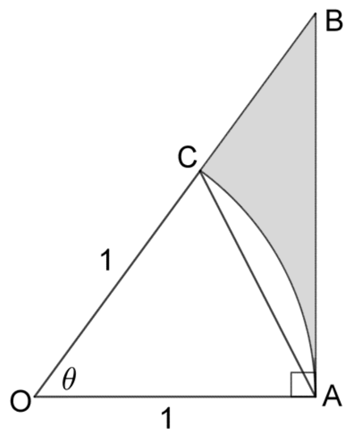
QUESTION 6 [25 MARKS]

6.1 The sketch shows the graph of $y = \arccos(x+1) - \frac{\pi}{3}$.



Determine the coordinates of the end points, A and B, as well as the x-value of C, the x-intercept of the graph. (6)

6.2 O is the centre of the sector OAC below with radius 1 and $C\hat{O}A = \theta$ with $\theta < \frac{\pi}{2}$.
The line perpendicular to radius OA intercepts OC extended in B and chord AC is drawn.



(a) By refering to the areas of the respective triangles and sector, arrange $\theta, \tan\theta$ and $\sin\theta$ in decending order. (5)

(b) If $\theta = \frac{\pi}{3}$, determine the circumference of the shaded area ABC. (5)

6.3 Given $f(x) = \begin{cases} x+4 & if\ x < -2 \\ -x-2 & if -2 \leq x < 1 \\ x^2 - 4x & if\ x \geq 1 \end{cases}$

Use the definition and determine the continuity of f at the following points.
If discontinuous, give the type as well.

(a) (i) $x = -2$ (3)

(ii) $x = 1$ (3)

(b) is f differentiable at $x = 1$? Motivate fully. (3)

QUESTION 7 [18 MARKS]

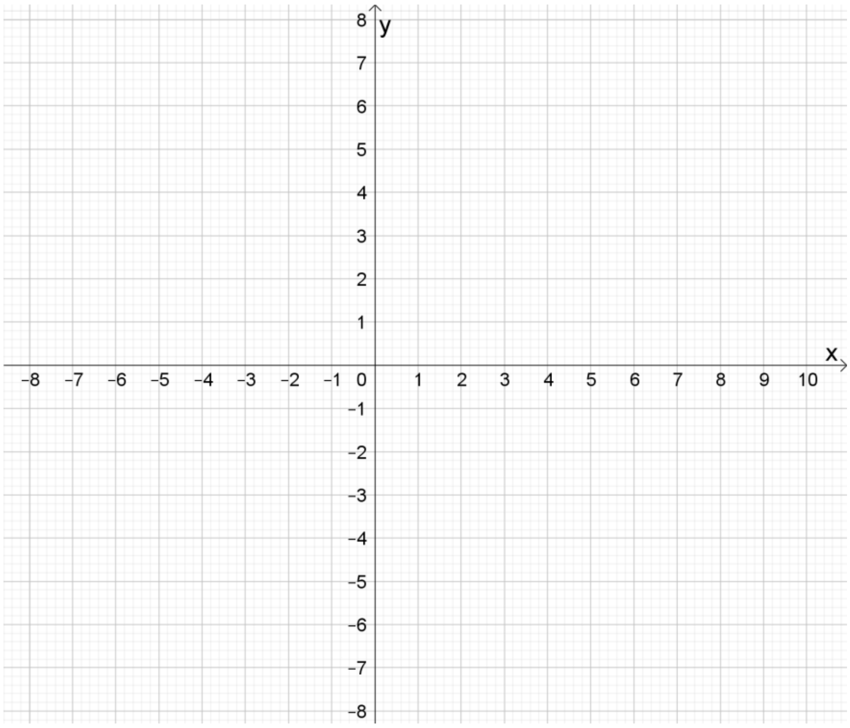
Given the graph with equation $\frac{x^2-3x+6}{1-x}$. **This graph does not intercept the x-axis**.

7.1 Determine the equations of all asymptotes of f. (4)

7.2 Determine the coordinates of the turning points of f, if any. (7)

7.3 Write down the y-intercept of f. (1)

7.4 Use the diagram sheet below and make a sketch graph of f.
Clearly show the coordinates of the turning points (if any), the asymptotes and the y-intercept on your sketch. (6)



QUESTION 8 [25 MARKS]

8.1 Differentiate the following functions as asked:

(a) If $f(x) = 2^{2x+1} + \mathrm{cosec}(x)$, determine $f'(x)$. (3)

(b) $D_x[\arcsin(ln(4x))]$ (3)

8.2 Given: $lny^2 - \log(xy) = \pi$ with $x \neq 0$ and $y \neq 0$.

(a) Expand the expression by using the laws of logarithms. (2)

(b) Hence or otherwise use implicit differentiation and determine $\frac{dy}{dx}$ (6)

8.3 The equation of a graph is $y = e^{-2x} \times \tan x$.

(a) Determine the value of x where the function has a stationary point if $-\frac{\pi}{2} < x < \frac{\pi}{2}$. (8)

(b) How can it be shown that this value of x will also give a point of inflection? Describe the process in words, do not do the mathematics. (3)

QUESTION 9 [19 MARKS]

Determine the following integrals:

9.1 $\int(2x-1)\sin(x^2-x+1)\,dx$ (4)

9.2 $\int_0^{\frac{\pi}{2}}(2x-1)\sin(2x)\,dx$ by using integration by parts (factor integration). Clearly show all the integration steps that you use and give your final answer in terms of π. (7)

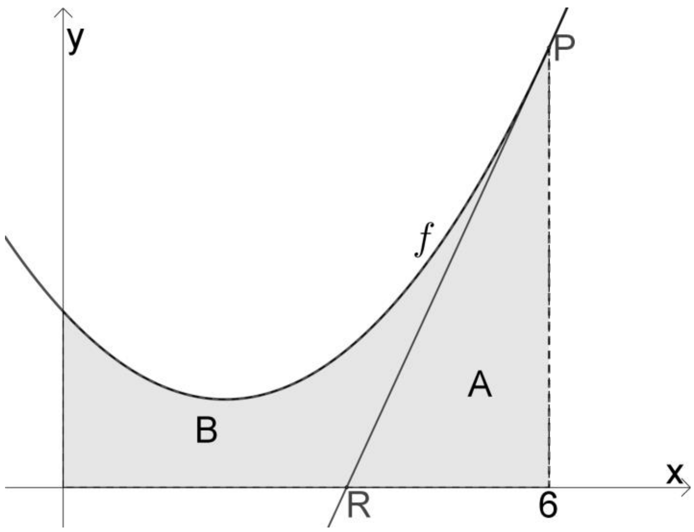
9.3 $\int \frac{5x-7}{(x-1)^2}\,dx$ by first decomposing the expression into partial fractions. (8)

QUESTION 10 [18 MARKS]

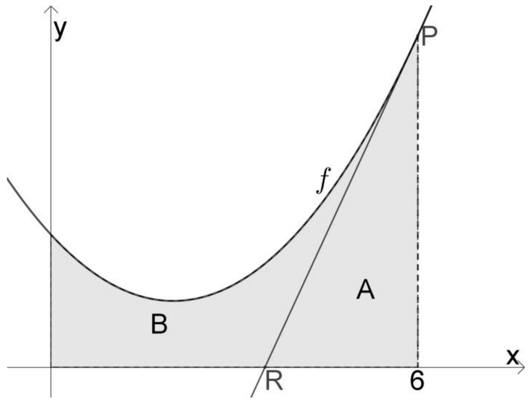
10.1 The graph of $\boldsymbol{f(x) = 1 + x\sin x}$ intercepts the x-axis in the interval [2;4]. Use Newton's method with $x = 3$ as first approximation and determine this point of interception correct to three decimal digits. (5)

10.2 The sketch beneath shows the graphs of $f(x) = x^2 - 4x + t$ (with t a constant) and the tangent PR to f at P.

- The x-value of P is 6 and R is the x-intercept of the tangent.
- The area between PR, the line $x = 6$ and the x-axis is A.
- The area between f, PR, the x-axis and the y-axis is B.
- The area of the shaded region A + B is 48.



(a) Determine the value of t. (5)



(b) Accept that $t = 8$ and determine the equation of PR. (5)

(c) The area B rotates around the x-axis. Write down the expression with which the volume of this rotating body can be calculated.
DO NOT CALCULATE IT. (3)

EXTRA SPACE