

α -MATHEMATICS

Alpha Mathematics FINAL EXAM PAPER

October 2024

Time: 3 hours

Grade 12

Total: 200 marks

INSTRUCTIONS AND INFORMATION

Carefully read through the following instructions before answering the examination paper:

1. Answer all 10 questions on this examination paper.
2. Write your name and ID number on the front page or the examination paper.
3. Non-programmable calculators may be used, unless otherwise indicated at a specific question.
4. Unless indicated otherwise, all answers, where applicable, must be given correct to two decimal places.
5. The diagrams in the examination paper are not necessarily drawn to scale.
6. All angles are given in radians. Answers must be given in radians where applicable.
7. This examination paper consists of a front page, 22 pages and a formula sheet.
8. Question 1 consists of **15** multiple choice questions. Answer it on the answer sheet. This answer sheet is at the front of the paper.
Do not remove the answer sheet.
9. For all other questions, all necessary calculations must be shown clearly. The correct answer on its own will not necessarily lead to full marks.
10. Additional writing space is provided at the end of this examination paper. Clearly indicate if you made use of this to complete a question.
11. Write neatly and legibly.

QUESTION 1**[30 MARKS]**

- 1.1 How many terms will the expansion of $x^2(2x - 5)^7$ have?
(A) 8 (B) 7 (C) 9 (D) 6
- 1.2 The power series in the expansion of $\sqrt{16 - 4x}$ will converge if:
(A) $|x| < \frac{1}{2}$ (B) $|x| > 2$ (C) $|x| < 2$ (D) $|x| < 4$
- 1.3 If $|1 - x| < 5$, then:
(A) $x < -6$ or $x > 4$ (B) $x < -4$ or $x > 6$
(C) $-6 < x < 4$ (D) $-4 < x < 6$
- 1.4 Simplify: $e^{2 \ln x}$
(A) $2x$ (B) x^2 (C) 2 (D) x
- 1.5 If $x - \sqrt{3} + 1$ is a factor of $f(x)$, which one of the following will also be a factor of $f(x)$?
(A) $x - 1 + \sqrt{3}$ (B) $x + 1 + \sqrt{3}$
(C) $x + 1 - \sqrt{3}$ (D) $x - 1 - \sqrt{3}$
- 1.6 How many roots does the equation $x^4 = 16$ have in $\mathbb{C}[x]$?
(A) 4 (B) 2 (C) 1 (D) 3
- 1.7 The graph of the function $f(x) = 2\arccos(x) - \frac{\pi}{2}$ intercepts the x -axis at $x = \dots$
(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{1}{2}$
- 1.8 The graph of the function $f(x) = \frac{ax^2 + bx + 2}{x - 3}$ has an oblique asymptote $y = -x + 1$.
Then $b = \dots$
(A) -1 (B) 1 (C) -2 (D) 4
- 1.9 If p is a polynomial of degree $n, n > 0$, what will the degree of $\int_0^x (p(t))^2 dt$ be?
(A) $n + 1$ (B) $n^2 + 1$ (C) $2n + 1$ (D) $2n - 1$

1.10 If $f(x) = \sin x$ and $f^n(x)$ is the n^{th} derivative of f relating to x , then the smallest value of n for which $f^n(x) = f(x)$ will be

- (A) 2 (B) 3 (C) 4 (D) 5

1.11 The functions f and g are both differentiable for all $x \in \mathbb{R}$. It is given that $g(x) > 0$ for all x . If $h(x) = f(x) \cdot g(x)$ and $h'(x) = f(x)g'(x)$, then $f(x) = \dots$

- (A) $\ln x$ (B) e^x (C) $\frac{1}{g(x)}$ (D) e

1.12 Given
$$\begin{cases} \frac{11-x}{2} + b(x) & x < 5 \\ \sqrt{x-1} & x \geq 5 \end{cases}$$

The function $f(x)$ should be modified as following so that it will be continuous at $x = 5$. A possibility for $b(x)$ is:

- (A) $\sqrt{x-1} - \frac{x-11}{2}$ (B) $\sqrt{x-1} + \frac{x-11}{2}$
 (C) $-3x + \sqrt{6}$ (D) $\sqrt{6x} - 3$

1.13 If n is a positive integer, then $\int_0^1 x^n dx = \int_0^1 (1-x)^n dx$ for

- (A) all n (B) all even n
 (C) all uneven n (D) no n

1.14 The area between the graph $y = \frac{4}{x}$, the x -axis and the line $x = m$ and $x = 2m$ with $m > 0$ will always

- (A) decrease as m increase (B) increase as m increase
 (C) be independent of m (D) increase if $m < \frac{1}{2}$, otherwise decrease

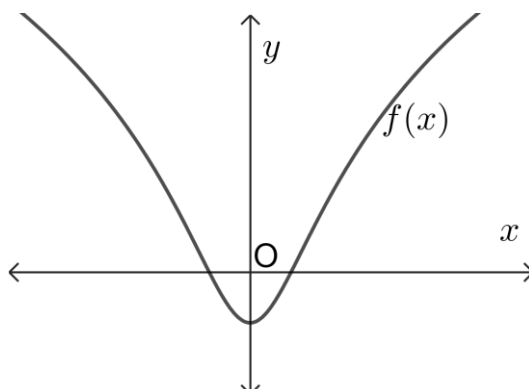
1.15 The functions $f(x)$ and $g(x)$ are differentiable for all values of x . The function g is the **inverse** of f with $g(-2) = 5$ and $f'(5) = \frac{1}{2}$. Then $g'(-2) =$

- (A) 2 (B) -2 (C) $\frac{1}{5}$ (D) $-\frac{1}{5}$

QUESTION 2**[16 MARKS]**

2.1 Solve for x : $2 \ln(x^2 + 1) - 1 = 0$. Give the answer(s) correct to **ONE** decimal digit. (4)

2.2 The sketch beneath shows the graph of $f(x) = 2 \ln(x^2 + 1) - 1$.



(a) Calculate the y -intercept of f and show it on the sketch. (2)

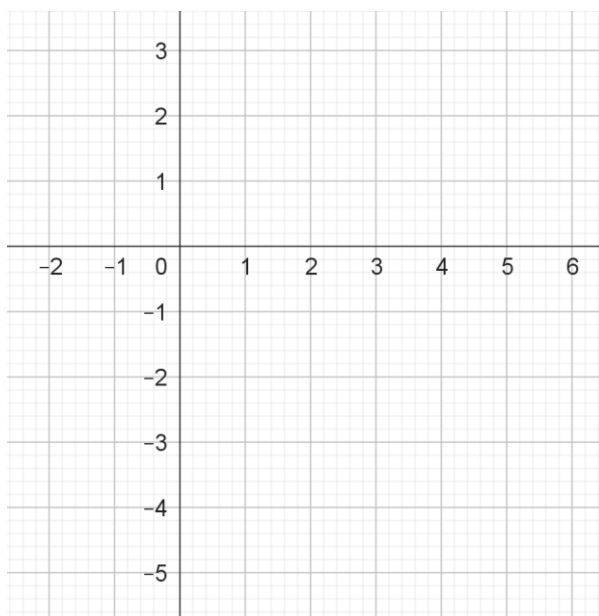
(b) Show your answer(s) determined in QUESTION 2.1 on the sketch above. (1)

(c) The domain of f is now restricted to $x \leq 0$.

Determine the equation of f^{-1} , the inverse of this restricted function. (4)

(d) Give the x - and y - intercept(s) of f^{-1} . (2)

(e) Sketch the graph of f^{-1} on the axes given below. Use your answers determined in QUESTION 2.2 (d) and show the intercepts with the axes clearly on the sketch. (3)

**QUESTION 3****[18 MARKS]**

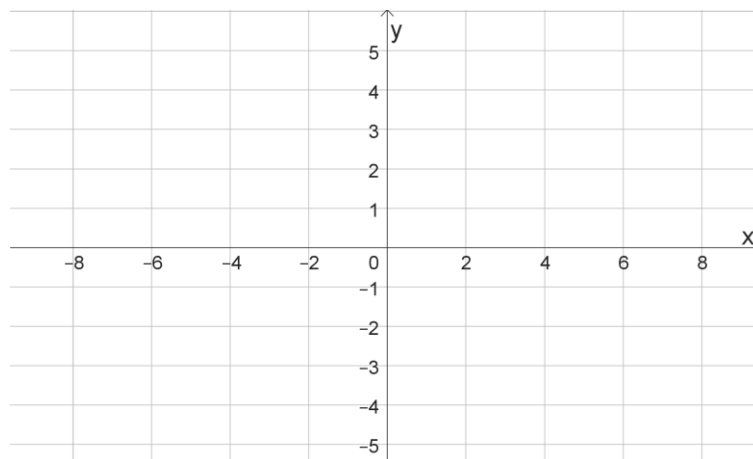
3.1 Determine the value of $\frac{(-1-i)^4}{-\sqrt{3}+i}$. Do the calculation in polar form and use π and root form if necessary. Leave the answer in polar form. (6)

3.2 Solve for x : $x^3 = -1$. Do the calculations in polar form. Give **all** the answers in **rectangular form**. (7)

- 3.3 Given that $x - i$ is a factor of $P(x) = x^4 - x^3 - 5x^2 - x - 6$.
Factorise $P(x)$ fully in the complex number system. (5)

QUESTION 4**[17 MARKS]**

- 4.1 Given $g(x) = |x - 2| - 4$
- (a) Write down the salient point of g . (1)
- (b) Determine all the intercepts of g with the axes. (4)
- (c) Hence sketch the graph of $y = |g(x)|$. Clearly show the intercepts with the axes and the salient point(s). (4)



- 4.2 Determine if the following set of equations has a unique solution. **It is not necessary to solve the set of equations.** Clearly show the matrix used with Cramer's method.

$$\begin{aligned} x + 3y + 2z &= 16 \\ x - y - z &= -10 \\ 3x + 5y + z &= 22 \end{aligned}$$

(3)

- 4.3 Use the binomial theorem and determine the constant in the expansion of:

$$\left(2x - \frac{5}{4x^2}\right)^9$$

(5)

QUESTION 5**[23 MARKS]**

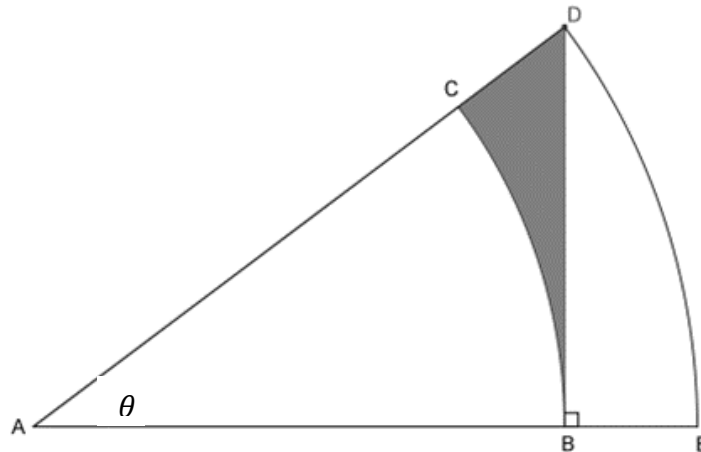
- 5.1 Given a triangle ABC , in a 3-dimensional space, with $A = (-1; 0; 1)$, $B = (-2; 4; -1)$ and $C = (1; 2; 0)$.
- (a) Write down the vectors \overrightarrow{AC} and \overrightarrow{BC} . (2)
- (b) Determine the value of \widehat{ACB} by using vector algebra. (5)
- 5.2 If \vec{u} and \vec{v} are algebraic vectors and a is a scalar (number), the identity $a(\vec{u} \times \vec{v}) = a\vec{u} \times \vec{v}$ is true (\times represents the cross product).
Given vectors $\vec{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\vec{v} = -\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.
Show that the identity is true by determining the value of $(3\vec{u}) \times \vec{v}$, if it is also given that $3(\vec{u} \times \vec{v}) = (-48; -15; 27)$. (4)

5.3 Given $y = 4 \arctan(x + 1) + \pi$.

Draw a sketch graph of the function and clearly show the intercepts with the axes as well as the asymptotes of the function on the sketch graph. Leave intercepts in terms of π where necessary. (6)

5.4 The sketch below shows two circle sectors, ABC and AED, with the same centre, A. The angle between the radii at A is θ .

$AE = 20$, $AB = 10\sqrt{3}$ and $DB \perp ABE$.



(a) Show that $\theta = \frac{\pi}{6}$ rad. (1)

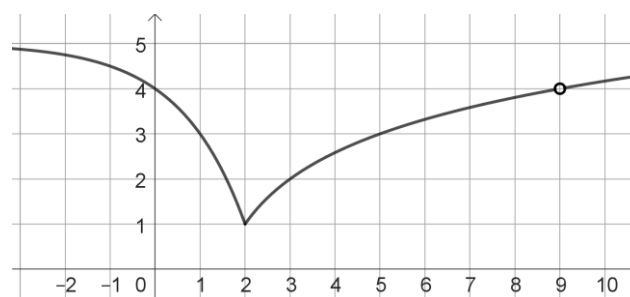
(b) Determine a and b if the area of the shaded region $CBD = a + b\pi$. (Give a and b in the simplest surd form if necessary.) (5)

QUESTION 6

[18 MARKS]

6.1 The sketch below shows the graph of

$$f(x) = \begin{cases} -2^x + 5 & \text{if } x \leq 2 \\ \log_2(x-1) + 1 & \text{if } x > 2, \quad x \neq 9 \end{cases}$$



(a) Give, with reason(s), the point(s) where f is discontinuous. Also give the type(s) of discontinuity. (3)

(b) Use the sketch and give the point(s) where f is not differentiable. (2)

(c) Motivate your answers of QUESTION.6.1 (b) fully algebraically. (5)

6.2 Prove the geometric sequence formula,

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \text{ for all } n \in \mathbb{N}$$

by using mathematical induction. (8)

QUESTION 7**[20 MARKS]**

7.1 Determine the derivative of the following as asked:

(a) Determine $\frac{d}{dx}[e^{x^2+\cos x}]$ (3)

(b) Determine $g'(x)$ if $g(x) = \ln(x^2 - 10x + 25)$. Simplify your answer. (3)

7.2 Use implicit differentiation and determine the equation of the tangent to $y^3 - xy = y + x^2$ at the point $(1; -1)$. (6)7.3 Look at the function $f(x) = x^2 - \sqrt{4+x}$, $x > -4$. This function is continuous on the given interval.

(a) Show that the function will have a zero in the interval $x \in [1; 2]$. (2)

(b) Use Newton-Rhapson method and determine this zero.

- Use $a_0 = 2$ as first approximation.
- Show the formula with which you work.
- Write down the second approximation (a_1) correct to 3 decimal digits.
- Give the answer rounded to 5 decimal digits. (6)

QUESTION 8**[19 MARKS]**8.1 Use a Riemann Sum and determine the value of $\int_0^3 (-x^2 + 6x) dx$ (9)8.2 Given the **derivative** of a function f , $f'(x) = 2(x+1)^2(2x-1) = 4x^3 + 6x^2 - 2$.(a) The function f passes through the point $(0; -1)$. Determine the equation of f . (3)(b) Determine x if $f'(x) = 0$ (1)(c) Determine x if $f''(x) = 0$ (2)(d) The graph of f has a stationary point that is not a point of inflection. Give the x -value of this point. (1)(e) The graph of f has a stationary point that is also a point of inflection. Give the x -value of this point and show that it is indeed a point of inflection. (3)**QUESTION 9****[19 MARKS]**

Determine the following integrals:

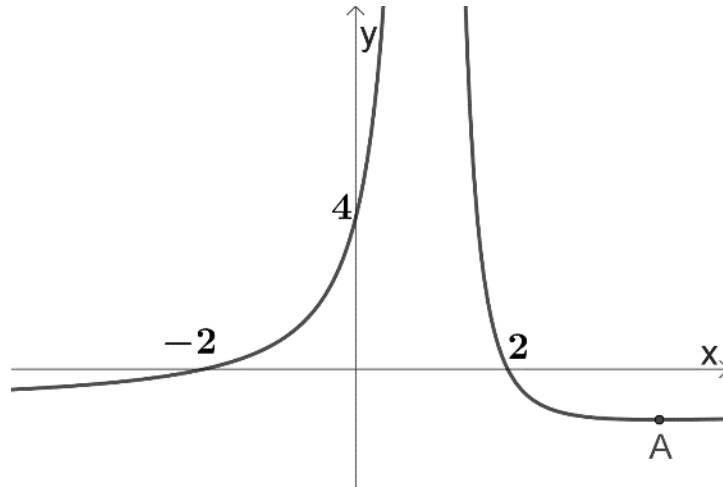
9.1 $\int (\sec^2(4x+1) + \operatorname{cosec} 3x \cot 3x + \pi) dx$ (5)

9.2 $\int \frac{8x^2-x-2}{(4x^2+1)(x+4)} dx$ by using partial fractions. (8)

9.3 $\int (x \cdot \cos^2(x)) dx$, by using factor (piece wise) integration. (6)

QUESTION 10**[20 MARKS]**

10.1 The sketch below shows the graph of $f(x) = \frac{4-x^2}{(x-1)^2}$.



Calculate the following and show it on the sketch if asked:

- The graph has a minimum turning point at A. Determine the coordinates of A. (5)
 - The graph intersects the horizontal asymptote, $y = -1$. Determine the x -value of this intersection. (3)
 - Write down the equation(s) of any other asymptotes of f . (1)
 - Do the following on the sketch given above:
Draw the asymptotes and show the value of the point calculated in QUESTION 10.1 (b) with a B. (2)
 - Write down the **equation** that can be used to calculate the volume of the body of revolution that is formed when f rotates around the x -axis with $x \in [2; 4]$. (You don't have to calculate the volume.) (2)
- 10.2 The area between the graph $\frac{a+\ln x}{x}$, the x -axis and the lines $x = 1$ and $x = e$ is equal to 3,5. Determine the value(s) of a . (7)